27

Factorial Experimental Designs

KEY WORDS *additivity, cube plot, density, design matrix, effect, factor, fly ash, factorial design, interaction, main effect, model matrix, normal order scores, normal plot, orthogonal, permeability, randomization, rankits, two-level design.*

Experiments are performed to (1) screen a set of factors (independent variables) and learn which produce an effect, (2) estimate the magnitude of effects produced by changing the experimental factors, (3) develop an empirical model, and (4) develop a mechanistic model. Factorial experimental designs are efficient tools for meeting the first two objectives. Many times, they are also excellent for objective three and, at times, they can provide a useful strategy for building mechanistic models.

Factorial designs allow a large number of variables to be investigated in few experimental runs. They have the additional advantage that no complicated calculations are needed to analyze the data produced. In fact, important effects are sometimes apparent without any calculations. The efficiency stems from using settings of the independent variables that are completely uncorrelated with each other. In mathematical terms, the experimental designs are *orthogonal*. The consequence of the orthogonal design is that the main effect of each experimental factor, and also the interactions between factors, can be estimated independent of the other effects.

Case Study: Compaction of Fly Ash

There was a proposal to use pozzolanic fly ash from a large coal-fired electric generating plant to build impermeable liners for storage lagoons and landfills. Pozzolanic fly ash reacts with water and sets into a rock-like material. With proper compaction this material can be made very impermeable. A typical criterion is that the liner must have a permeability of no more than 10^{-7} cm/sec. This is easily achieved using small quantities of fly ash in the laboratory, but in the field there are difficulties because the rapid pozzolanic chemical reaction can start to set the fly ash mixture before it is properly compacted. If this happens, the permeability will probably exceed the target of 10^{-7} cm/sec.

As a first step it was decided to study the importance of water content (%), compaction effort (psi), and reaction time (min) before compaction. These three factors were each investigated at two levels. This is a *two-level, three-factor experimental design*. Three factors at two levels gives a total of eight experimental conditions. The eight conditions are given in Table 27.1, where W denotes water content (4% or 10%), C denotes compaction effort (60 psi or 260 psi), and T denotes reaction time (5 or 20 min). Also given are the measured densities, in lb/ft^3. The permeability of each test specimen was also measured. The data are not presented, but permeability was inversely proportional to density. The eight test specimens were made at the same time and the eight permeability tests started simultaneously (Edil et al., 1987).

The results of the experiment are presented as a cube plot in Figure 27.1. Each corner of the cube represents one experimental condition. The plus (+) and minus (–) signs indicate the levels of the factors. The top of the cube represents the four tests at high compression, whereas the bottom represents the four tests at low pressure. The front of the cube shows the four tests at low reaction time, while the back shows long reaction time.

It is apparent without any calculations that each of the three factors has some effect on density. Of the investigated conditions, the best is run 4 with high water content, high compaction effort, and short

TABLE 27.1

Experimental Conditions and Responses for Eight Fly Ash Specimens

Run	Factor W (%)	C (psi)	T (min)	Density (lb/ft^3)
1	4	60	5	107.9
2	10	60	5	120.8
3	4	260	5	118.6
4	10	260	5	126.5
5	4	60	20	99.8
6	10	60	20	117.5
7	4	260	20	107.6
8	10	260	20	118.9

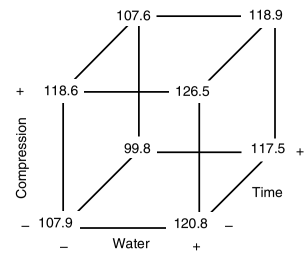

FIGURE 27.1 Cube plot showing the measured densities for the eight experimental conditions of the 2^3 factorial design.

reaction time. Densities are higher at the top of the cube than at the bottom, showing that higher pressure increases density. Density is lower at the back of the cube than at the front, showing that long reaction time reduces density. Higher water content increases density. The difference between the response at high and low levels is called a *main effect*. They can be quantified and tested for statistical significance.

It is possible that density is affected by how the factors act in combination. For example, the effect of water content at 20-min reaction time may not be the same as at 5 min. If it is not, there is said to be a *two-factor interaction* between water content and reaction time. Water content and compaction might interact, as might compaction and time.

Method: A Full 2^k Factorial Design

The k independent variables whose possible influence on a response variable is to be assessed are referred to as factors. An experiment with k factors, each set at two levels, is called a *two-level factorial design*. A *full factorial design* involves making runs at 2^k different experimental conditions which represent all combinations of the k factors at high and low levels. This is also called a *saturated design*. The high and low levels are conveniently denoted by + and –, or by +1 and –1. The factors can be continuous (pressure, temperature, concentration, etc.) or discrete (additive present, source of raw material, stirring used, etc.) The response variable (dependent variable) is y.

There are two-level designs that use less than 2^k runs to investigate k factors. These *fractional factorial designs* are discussed in Chapter 28. An experiment in which each factor is set at three levels would be a three-level factorial design (Box and Draper, 1987; Davies, 1960). Only two-level designs will be considered here.

TABLE 27.2

Design Matrices for 2^3 and 2^4 Full Factorial Designs

Run Number	Factor 1	Factor 2	Factor 3	Run Number	Factor 1	Factor 2	Factor 3	Factor 4
1	−	−	−	1	−	−	−	−
2	+	−	−	2	+	−	−	−
3	−	+	−	3	−	+	−	−
4	+	+	−	4	+	+	−	−
5	−	−	+	5	−	−	+	−
6	+	−	+	6	+	−	+	−
7	−	+	+	7	−	+	+	−
8	+	+	+	8	+	+	+	−
				9	−	−	−	+
				10	+	−	−	+
				11	−	+	−	+
				12	+	+	−	+
				13	−	−	+	+
				14	+	−	+	+
				15	−	+	+	+
				16	+	+	+	+

Experimental Design

The *design matrix* lists the setting of each factor in a standard order. Table 27.2 contains the design matrix for a full factorial design with $k = 3$ factors at two levels and a $k = 4$ factor design. The three-factor design uses $2^3 = 8$ experimental runs to investigate three factors. The 2^4 design uses 16 runs to investigate four factors. Note the efficiency: only 8 runs to investigate three factors, or 16 runs to investigate four factors.

The design matrix provides the information needed to set up each experimental test condition. Run number 5 in the 2^3 design, for example, is to be conducted with factor 1 at its low (−) setting, factor 2 at its low (−) setting, and factor 3 at its high (+) setting. If all the runs cannot be done simultaneously, they should carried out in *randomized* order to avoid the possibility that unknown or uncontrolled changes in experimental conditions might bias the factor effect. For example, a gradual increase in response over time might wrongly be attributed to factor 3 if runs were carried out in the standard order sequence. The lower responses would occur in the early runs where 3 is at the low setting, while the higher responses would tend to coincide with the + settings of factor 3.

Data Analysis

The statistical analysis consists of estimating the effects of the factors and assessing their significance. For a 2^3 experiment we can use the cube plots in Figure 27.2 to illustrate the nature of the estimates of the three main effects.

The main effect of a factor measures the average change in the response caused by changing that factor from its low to its high setting. This experimental design gives four separate estimates of each effect. Table 27.2 shows that the only difference between runs 1 and 2 is the level of factor 1. Therefore, the difference in the response measured in these two runs is an estimate of the effect of factor 1. Likewise, the effect of factor 1 is estimated by comparing runs 3 and 4, runs 5 and 6, and runs 7 and 8. These four estimates of the effect are averaged to estimate the main effect of factor 1.

This can also be shown graphically. The main effect of factor 1, shown in panel a of Figure 27.2, is the average of the responses measured where factor 1 is at its high (+) setting minus the average of the low (−) setting responses. Graphically, the average of the four corners with small dots are subtracted from the average of the four corners with large dots. Similarly, the main effects of factor 2 (panel b) and factor 3 (panel c) are the differences between the average at the high settings and the low settings for factors 2 and 3. Note that the effects are the changes in the response resulting from changing a factor from the low to the high level. It is not, as we are accustomed to seeing in regression models, the change associated with a one-unit change in the level of the factor.

TABLE 27.3

Model Matrix for a 2^3 Full Factorial Design

Run	X_0	X_1	X_2	X_3	X_{12}	X_{13}	X_{23}	X_{123}	y
1	+1	−1	−1	−1	+1	+1	+1	−1	y_1
2	+1	+1	−1	−1	−1	−1	+1	+1	y_2
3	+1	−1	+1	−1	−1	+1	−1	+1	y_3
4	+1	+1	+1	−1	+1	−1	−1	−1	y_4
5	+1	−1	−1	+1	+1	−1	−1	+1	y_5
6	+1	+1	−1	+1	−1	+1	−1	−1	y_6
7	+1	−1	+1	+1	−1	−1	+1	−1	y_7
8	+1	+1	+1	+1	+1	+1	+1	+1	y_8

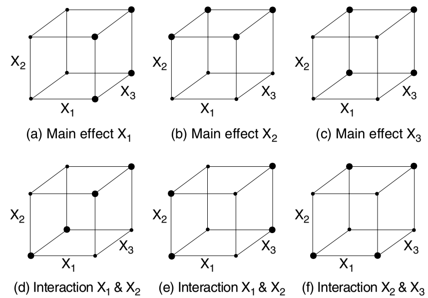

FIGURE 27.2 Cube plots showing the main effects and two-factor interactions of a 2^3 factorial experimental design. The main effects and interactions are estimated by subtracting the average of the four values indicated with small dots from the average of the four values indicated by large dots.

The interactions measure the *non-additivity* of the effects of two or more factors. A significant *two-factor interaction* indicates antagonism or synergism between two factors; their combined effect is not the sum of their separate contributions. The interaction between factors 1 and 2 (panel d) is the average difference between the effect of factor 1 at the high setting of factor 2 and the effect of factor 1 at the low setting of factor 2. Equivalently, it is the effect of factor 2 at the high setting of factor 1 minus the effect of factor 2 at the low setting of factor 1. This interpretation holds for the two-factor interactions between factors 1 and 3 (panel e) and factors 2 and 3 (panel f). This is equivalent to subtracting the average of the four corners with small dots from the average of the four corners with large dots.

There is also a three-factor interaction. Ordinarily, this is expected to be small compared to the two factor interactions and the main effects. This is not diagrammed in Figure 27.2.

The effects are estimated using the *model matrix*, shown in Table 27.3. The structure of the matrix is determined by the model being fitted to the data. The model to be considered here is linear and it consists of the average plus three main effects (one for each factor) plus three two-factor interactions and a three-factor interaction. The model matrix gives the signs that are used to calculate the effects.

This model matrix consists of a column vector for the average, plus one column for each main effect, one column for each interaction effect, and a column vector of the response values. The number of columns is equal to the number of experimental runs because eight runs allow eight parameters to be estimated. The elements of the column vectors (X_i) can always be coded to be +1 or −1, and the signs are determined from the design matrix, Table 27.3. X_0 is always a vector of +1. X_1 has the signs associated with factor 1 in the design matrix, X_2 those associated with factor 2, and X_3 those of factor 3, etc. for higher-order full factorial designs. These vectors are used to estimate the main effects.

Interactions are represented in the model matrix by cross-products. The elements in X_{12} are the products of X_1 and X_2 (for example, (–1)(–1) = 1, (1)(–1) = –1, (–1)(1) = –1, (1)(1) = 1, etc.). Similarly, X_{13} is X_1 times X_3. X_{23} is X_2 times X_3. Likewise, X_{123} is found by multiplying the elements of X_1, X_2, and X_3 (or the equivalent, X_{12} times X_3, or X_{13} times X_2). The order of the X vectors in the model matrix is not important, but the order shown (a column of +1's, the factors, the two-factor interactions, followed by higher-order interactions) is a standard and convenient form.

From the eight response measurements $y_1, y_2, \ldots, y_8$, we can form eight statistically independent quantities by multiplying the $\boldsymbol{y}$ vector by each of the $\boldsymbol{X}$ vectors. The reason these eight quantities are statistically independent derives from the fact that the X vectors are orthogonal.[1] The independence of the estimated effects is a consequence of the orthogonal arrangement of the experimental design.

This multiplication is done by applying the signs of the $\boldsymbol{X}$ vector to the responses in the $\boldsymbol{y}$ vector and then adding the signed y's. For example, $\boldsymbol{y}$ multiplied by $\boldsymbol{X}_0$ gives the sum of the responses: $\boldsymbol{X}_0 \cdot \boldsymbol{y} = y_1 + y_2 + \cdots + y_8$. Dividing the quantity $\boldsymbol{X}_0 \cdot \boldsymbol{y}$ by 8 gives the average response of the whole experiment. Multiplying the $\boldsymbol{y}$ vector by an $\boldsymbol{X}_i$ vector yields the sum of the four differences between the four y's at the +1 levels and the four y's at the –1 levels. The effect is estimated by the average of the four differences; that is, the effect of factor $\boldsymbol{X}_i$ is $\boldsymbol{X}_i \cdot \boldsymbol{y}/4$.

The eight effects and interactions that can be calculated from a full eight-run factorial design are:

Average $$\boldsymbol{X}_0 \cdot \boldsymbol{y} = \frac{y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8}{8}$$

Main effect of factor 1 $$\boldsymbol{X}_1 \cdot \boldsymbol{y} = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{4}$$

$$= \frac{y_2 + y_4 + y_6 + y_8}{4} - \frac{y_1 + y_3 + y_5 + y_7}{4}$$

Main effect of factor 2 $$\boldsymbol{X}_2 \cdot \boldsymbol{y} = \frac{y_3 + y_4 + y_7 + y_8}{4} - \frac{y_1 + y_2 + y_5 + y_6}{4}$$

Main effect of factor 3 $$\boldsymbol{X}_3 \cdot \boldsymbol{y} = \frac{y_5 + y_6 + y_7 + y_8}{4} - \frac{y_1 + y_2 + y_3 + y_4}{4}$$

Interaction of factors 1 and 2 $$\boldsymbol{X}_{12} \cdot \boldsymbol{y} = \frac{y_1 + y_4 + y_5 + y_8}{4} - \frac{y_2 + y_3 + y_6 + y_7}{4}$$

Interaction factors 1 and 3 $$\boldsymbol{X}_{13} \cdot \boldsymbol{y} = \frac{y_1 + y_3 + y_6 + y_8}{4} - \frac{y_2 + y_4 + y_5 + y_7}{4}$$

Interaction of factors 2 and 3 $$\boldsymbol{X}_{23} \cdot \boldsymbol{y} = \frac{y_1 + y_2 + y_7 + y_8}{4} - \frac{y_3 + y_4 + y_5 + y_6}{4}$$

Interaction of factors 1, 2, and 3 $$\boldsymbol{X}_{123} \cdot \boldsymbol{y} = \frac{y_2 + y_3 + y_5 + y_8}{4} - \frac{y_1 + y_4 + y_6 + y_7}{4}$$

If the variance of the individual measurements is σ^2, the variance of the mean is:

$$\mathrm{Var}(\bar{y}) = \left(\frac{1}{8}\right)^2 [\mathrm{Var}(y_1) + \mathrm{Var}(y_2) + \cdots + \mathrm{Var}(y_8)] = \left(\frac{1}{8}\right)^2 8\sigma^2 = \frac{\sigma^2}{8}$$

The variance of each main effect and interaction is:

$$\mathrm{Var}(\text{effect}) = \left(\frac{1}{4}\right)^2 [\mathrm{Var}(y_1) + \mathrm{Var}(y_2) + \cdots + \mathrm{Var}(y_8)] = \left(\frac{1}{4}\right)^2 8\sigma^2 = \frac{\sigma^2}{2}$$

[1] Orthogonal means that the product of any two-column vectors is zero. For example, $\boldsymbol{X}_3 \cdot \boldsymbol{X}_{123} = (-1)(-1) + \cdots + (+1)(+1) = 1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 = 0$.

The experimental design just described does not produce an estimate of σ^2 because there is no replication at any experimental condition. In this case the significance of effects and interactions is determined from a normal plot of the effects (Box et al., 1978). This plot is illustrated later.

Case Study Solution

The responses at each setting and the calculation of the main effects are shown on the cube plots in Figure 27.3. As in Figure 27.1, each corner of the cube is the density measured at one of the eight experimental conditions.

The average density is $(\boldsymbol{X}_0 \cdot \boldsymbol{y})$:

$$\frac{107.9 + 120.8 + 118.6 + 126.5 + 99.8 + 117.5 + 107.6 + 118.9}{8} = 114.7$$

The estimates of the three main effects, the three two-factor interactions, and the one three-factor interaction are:

Main effect of water $(\boldsymbol{X}_1 \cdot \boldsymbol{y})$

$$\frac{120.8 + 126.5 + 117.5 + 118.9}{4} - \frac{107.9 + 118.6 + 99.8 + 107.6}{4} = 12.45$$

Main effect of compaction $(\boldsymbol{X}_2 \cdot \boldsymbol{y})$

$$\frac{118.6 + 126.5 + 107.6 + 118.9}{4} - \frac{107.9 + 120.8 + 99.8 + 117.5}{4} = 6.40$$

Main effect of time $(\boldsymbol{X}_3 \cdot \boldsymbol{y})$

$$\frac{99.8 + 117.5 + 107.6 + 118.9}{4} - \frac{107.9 + 120.8 + 118.6 + 126.5}{4} = -7.50$$

Two-factor interaction of water × compaction $(\boldsymbol{X}_{12} \cdot \boldsymbol{y})$

$$\frac{107.9 + 126.5 + 99.8 + 118.9}{4} - \frac{120.8 + 118.6 + 117.5 + 107.6}{4} = -2.85$$

Two-factor interaction of water × time $(\boldsymbol{X}_{13} \cdot \boldsymbol{y})$

$$\frac{107.9 + 118.6 + 117.5 + 118.9}{4} - \frac{120.8 + 126.5 + 99.8 + 107.6}{4} = -2.05$$

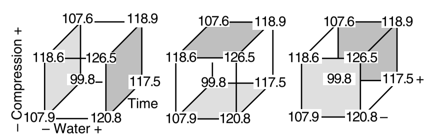

FIGURE 27.3 Cube plots of the 2^3 factorial experimental design. The values at the corners of the cube are the measured densities at the eight experimental conditions. The shaded faces indicate how the main effects are computed by subtracting the average of the four values at the low setting (– sign; light shading) from the average of the four values at the high setting (+ sign; dark shading).

Two-factor interaction of compaction × time ($X_{23} \cdot y$)

$$\frac{107.9 + 120.8 + 107.6 + 118.9}{4} - \frac{118.6 + 126.5 + 99.8 + 117.5}{4} = -1.80$$

Three-factor interaction of water × compaction × time ($X_{123} \cdot y$)

$$\frac{120.8 + 118.6 + 99.8 + 118.9}{4} - \frac{107.9 + 126.5 + 117.5 + 107.6}{4} = -0.35$$

Before interpreting these effects, we want to know whether they are large enough not to have arisen from random error. If we had an estimate of the variance of measurement error, the variance of each effect could be estimated and confidence intervals could be used to make this assessment. In this experiment there are no replicated measurements, so it is not possible to compute an estimate of the variance. Lacking a variance estimate, another approach is used to judge the significance of the effects.

If the effects are random (i.e., arising from random measurement errors), they might be expected to be normally distributed, just as other random variables are expected to be normally distributed. Random effects will plot as a straight line on normal probability paper. The *normal plot* is constructed by ordering the effects (excluding the average), computing the probability plotting points as shown in Chapter 5, and making a plot on normal probability paper. Because probability paper is not always handy, and many computer graphics programs do not make probability plots, it is handy to plot the effects against the *normal order scores* (or *rankits*). Table 27.4 shows both the probability plotting positions and the normal order scores for the effects.

Figure 27.4 is a plot of the estimated effects estimated against the normal order scores. Random effects will fall along a straight line on this plot. These are not *statistically significant*. We consider them to have values of zero. Nonrandom effects will fall off the line; these effects will be the largest (in absolute value). The nonrandom effects are considered to be statistically significant.

In this case a straight line covers the two- and three-factor interactions on the normal plot. None of the interactions are significant. The significant effects are the main effects of water content, compaction effort, and reaction time. Notice that it is possible to draw a straight line that covers the main effects and leaves the interactions off the line. Such an interpretation — significant interactions and insignificant main effects — is not physically plausible. Furthermore, effects of near-zero magnitude cannot be significant when effects with larger absolute values are not.

TABLE 27.4

Effects, Plotting Positions, and Normal Order Scores for Figure 27.4

Order number *i*	1	2	3	4	5	6	7
Identity of effect	3	12	23	123	13	2	1
Effect	−7.5	−2.85	−1.80	−0.35	2.05	6.40	12.45
$P = 100(i - 0.5)/7$	0.07	0.21	0.36	0.50	0.64	0.79	0.93
Normal order scores	−1.352	−0.757	−0.353	0	0.353	0.757	1.352

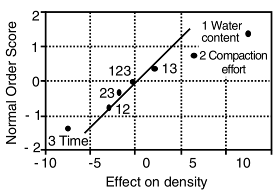

FIGURE 27.4 Normal probability plot of the estimated main effects and interactions.

The final interpretation of the results is:

1. The average density over the eight experimental design conditions is 114.7.
2. Increasing water content from 4 to 10% increases the density by an average of 12.45 lb/ft^3.
3. Increasing compaction effort from 60 to 260 psi increases density by an average of 6.40 lb/ft^3.
4. Increasing reaction time from 5 to 20 min decreases density by an average of 7.50 lb/ft^3.
5. These main effects are additive because the interactions are zero. Therefore, increasing both water content and compaction effort from their low to high values increases density by 12.45 + 6.40 = 18.85 lb/ft^3.

Comments

Two-level factorial experiments are a way of investigating a large number of variables with a minimum number of experiments. In general, a k variable two-level factorial experiment will require 2^k experimental runs. A 2^2 experiment evaluates two variables in four runs, a 2^3 experiment evaluates three variables in eight runs, a 2^4 design evaluates four variables in sixteen runs, etc. The designs are said to be *full* or *saturated*. From this small number of runs it is possible to estimate the average level of the response, k main effects, all two-factor interactions, and all higher-order interactions. Furthermore, these main effects and interactions are estimated independently of each other. Each main effect independently estimates the change associated with one experimental factor, and only one.

Why do so few experimental runs provide so much information? The strength and beauty of this design arise from its economy and balance. Each data point does triple duty (at least) in estimating main effects. Each observation is used in the computation of each factor main effect and each interaction. Main effects are averaged over more than one setting of the companion variables. This is the result of varying all experimental factors simultaneously. One-factor-at-a-time (OFAT) designs have none of this efficiency or power. An OFAT design in eight runs would provide only estimates of the main effects (no interactions) and the estimates of the main effects would be inferior to those of the two-level factorial design.

The statistical significance of the estimated effects can be evaluated by making the normal plot. If the effects represent only random variation, they will plot as a straight line. If a factor has caused an effect to be larger than expected due to random error alone, the effect will not fall on a straight line. Effects of this kind are interpreted as being significant. Another way to evaluate significance is to compute a confidence interval, or a reference distribution. This is shown in Chapter 28.

Factorial designs should be the backbone of an experimenter's design strategy. Chapter 28 shows how four factors can be evaluated with only eight runs. Experimental designs of this kind are called fractional factorials. Chapter 29 extends this idea. In Chapter 30 we show how the effects are estimated by linear algebra or regression, which is more convenient in larger designs and in experiments where the independent variables have not been set exactly according to the orthogonal design. Chapter 43 explains how factorial designs can be used sequentially to explore a process and optimize its performance.

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Exercises

27.1 Recycled Water Irrigation. Evaluate an irrigation system that uses recycled water to grow cucumbers and eggplant. Some field test data are given in the table below. Irrigation water was applied in two ways: sprinkle and drip. Evaluate the yield, yield per gallon, and biomass production

Vegetable	Irrigation Type	Irrigation Source	Yield (lb/ft^2)	Yield (lb/gal)	Biomass (lb/plant)
Cucumber	Sprinkle	Tap water	6.6	0.15	5.5
		Recycled water	6.6	0.15	5.7
	Drip	Tap water	4.9	0.25	4.5
		Recycled water	4.8	0.25	4.0
Eggplant	Sprinkle	Tap water	2.9	0.07	3.0
		Recycled water	3.2	0.07	3.5
	Drip	Tap water	1.6	0.08	1.9
		Recycled water	2.3	0.12	2.3

27.2 Water Pipe Corrosion. Students at Tufts University collected the following data to investigate the concentration of iron in drinking water as a means of inferring water pipe corrosion. (a) Estimate the main effects and interactions of the age of building, type of building, and location. (b) Make the normal plot to judge the significance of the estimated effects. (c) Based on duplicate observations at each condition, the estimate of σ is 0.03. Use this value to calculate the variance of the average and the main and interaction effects. Use $\text{Var}(\bar{y}) = S_p^2/N$ and $\text{Var}(\text{Effect}) = 4S_p^2/N$, where N = total number of measurements (in this case $N = 16$) to evaluate the results. Compare your conclusions regarding significance with those made using the normal plot.

Age	Type	Location	Iron (mg/L)	
Old	Academic	Medford	0.23	0.28
New	Academic	Medford	0.36	0.29
Old	Residential	Medford	0.03	0.06
New	Residential	Medford	0.05	0.02
Old	Academic	Somerville	0.08	0.05
New	Academic	Somerville	0.03	0.08
Old	Residential	Somerville	0.04	0.07
New	Residential	Somerville	0.02	0.06

27.3 Bacterial Tests. Analysts A and B each made bacterial tests on samples of sewage effluent and water from a clean stream. The bacterial cultures were grown on two media: M1 and M2. The experimental design is given below. Each test condition was run in triplicate. The y values are logarithms of the measured bacterial populations. The s_i^2 are the variances of the three replicates at each test condition. (a) Calculate the main and interaction effects using the averages at each test condition. (b) Draw the normal plot to interpret the results.(c) Average the eight variances to estimate σ^2 for the experiment. Use $\text{Var}(\bar{y}) = \frac{\sigma^2}{24}$ and $\text{Var}(\text{effect}) = \frac{\sigma^2}{6}$ to evaluate the results. [Note that Var(effect) applies to main effects *and* interactions. These variance equations account for the replication in the design.]

Source	Analyst	Medium	y (3 Replicates)			$\bar{y}_i$	s_i^2
Effluent	A	M1	3.54	3.79	3.40	3.58	0.0390
Stream	A	M1	1.85	1.76	1.72	1.78	0.0044
Effluent	B	M1	3.81	3.82	3.79	3.81	0.0002
Stream	B	M1	1.72	1.75	1.55	1.67	0.0116
Effluent	A	M2	3.63	3.67	3.71	3.67	0.0016
Stream	A	M2	1.60	1.74	1.72	1.69	0.0057
Effluent	B	M2	3.86	3.86	4.08	3.93	0.0161
Stream	B	M2	2.05	1.51	1.70	1.75	0.0750

27.4 Reaeration. The data below are from an experiment that attempted to relate the rate of dissolution of an organic chemical to the reaeration rate (y) in a laboratory model stream channel. The three experimental factors are stream velocity (V, in m/sec), stream depth (D, in cm), and channel roughness (R). Calculate the main effects and interactions and interpret the results.

Run	V	D	R	y (Triplicates)			Average
1	0.25	10	Smooth	107	117	117	113.7
2	0.5	10	Smooth	190	178	179	182.3
3	0.25	15	Smooth	119	116	133	122.7
4	0.5	15	Smooth	188	191	195	191.3
5	0.25	10	Coarse	119	132	126	125.7
6	0.5	10	Coarse	187	173	166	175.3
7	0.25	15	Coarse	140	133	132	135.0
8	0.5	15	Coarse	164	145	144	151.0

27.5 Metal Inhibition. The results of a two-level, four-factor experiment to study the effect of zinc (Zn), cobalt (Co), and antimony (Sb) on the oxygen uptake rate of activated sludge are given below. Calcium (Ca) was added to some test solutions. The (–) condition is absence of Ca, Zn, Co, or Sb. The (+) condition is 10 mg/L Zn, 1 mg/L Co, 1 mg/L Sb, or 300 mg/L Ca (as $CaCO_3$). The control condition (zero Ca, Zn, Co, and Sb) was duplicated. The measured response is cumulative oxygen uptake (mg/L) in 20-hr reaction time. Interpret the data in terms of the main and interaction effects of the four factors.

Run	Zn	Co	Sb	Ca	Uptake (mg/L)
1	−1	−1	−1	−1	761
2	+1	−1	−1	−1	532
3	−1	+1	−1	−1	759
4	+1	+1	−1	−1	380
5	−1	−1	+1	−1	708
6	+1	−1	+1	−1	348
7	−1	+1	+1	−1	547
8	+1	+1	+1	−1	305
9	−1	−1	−1	+1	857
10	+1	−1	−1	+1	902
11	−1	+1	−1	+1	640
12	+1	+1	−1	+1	636
13	−1	−1	+1	+1	822
14	+1	−1	+1	+1	798
15	−1	+1	+1	+1	511
16	+1	+1	+1	+1	527
1 (rep)	−1	−1	−1	−1	600

Source: Hartz, K. E., *J. WPFC,* 57, 942–947.

27.6 Plant Lead Uptake. Anaerobically digested sewage sludge and commercial fertilizer were applied to garden plots (10 ft × 10 ft) on which were grown turnips or Swiss chard. Each treatment was done in triplicate. After harvesting, the turnip roots or Swiss chard leaves were washed, dried, and analyzed for total lead. Determine the main and interaction effects of the sludge and fertilizer on lead uptake by these plants.

Exp.	Sludge	Fertilizer	Turnip Root	Swiss Chard Leaf
1	None	None	0.46, 0.57, 0.43	2.5, 2.7, 3.0
2	110 gal/plot	None	0.56, 0.53, 0.66	2.0, 1.9, 1.4
3	None	2.87 lb/plot	0.29, 0.39, 0.30	3.1, 2.5, 2.2
4	110 gal/plot	2.87 lb/plot	0.31, 0.32, 0.40	2.5, 1.6, 1.8

Source: Auclair, M. S. (1976). M.S. thesis, Civil Engr. Dept., Tufts University.