Spacetime models in `R-INLA`

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Outline

- Space-time modeling
  - Vector AR(1) example
  - Kronecker product models:
    - Separable space-time models
    - Discrete and continuous time
- PM-10 concentration in Piemonte, Italy

Space-time modeling

- spatial process that evolves over time

$$\boldsymbol{x} = \{x_{11}, ..., x_{n1}, x_{12}, ..., x_{nT}\}$$

- model the entire $\boldsymbol{x}$ at once

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- Simplest way: Separable precision/covariance
  - take a two-dimensional model for space
  - take a univariate model for time
  - 'combine it' (next slide example)

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- Simplest way: Separable precision/covariance
  - take a two-dimensional model for space
  - take a univariate model for time
  - 'combine it' (next slide example)
- Another way:
  - non-separable precision/covariance

Vector AR(1) process

- $\boldsymbol{x} = \{x_{11}, ..., x_{n1}, x_{12}, ..., x_{nT}\}$
- vector AR(1) process: $\boldsymbol{x}_t = \{x_{1t}, ..., x_{nt}\}$

$$\boldsymbol{x}_t = \rho \boldsymbol{x}_{t-1} + \boldsymbol{\omega}_t$$

- $\boldsymbol{\omega_t}$: correlated error innovation every time

$$\boldsymbol{\omega}_t \overset{\text{i.i.d.}}{\sim} N(\mathbf{0}, \boldsymbol{Q}^{-1}),$$

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- ρ: temporal correlation
- space-time: when $\boldsymbol{\omega}_t$ follows some spatial model
- a AR(1) process for every time series

From [Cameletti et al., 2012]

$$\boldsymbol{\xi}_t = a\boldsymbol{\xi}_{t-1} + \tilde{\boldsymbol{\omega}}_t, \quad \tilde{\boldsymbol{\omega}}_t \sim N\left(\mathbf{0}, \mathbf{Q}_S^{-1}\right) \tag{10}$$

for $t = 1, \ldots, T$ and with $\boldsymbol{\xi}_1 \sim N(\mathbf{0}, \mathbf{Q}_S^{-1}/(1-a^2))$. It follows that the joint distribution of the Tn-dimensional GMRF $\boldsymbol{\xi} = (\boldsymbol{\xi}_1', \ldots, \boldsymbol{\xi}_T')'$ is

$$\boldsymbol{\xi} \sim N\left(\mathbf{0}, \mathbf{Q}^{-1}\right) \tag{11}$$

with $\mathbf{Q} = \mathbf{Q}_T \otimes \mathbf{Q}_S$ where

$$\mathbf{Q}_T = \begin{pmatrix} 1/\sigma_\omega^2 & -a/\sigma_\omega^2 & & & \\ -a/\sigma_\omega^2 & (1+a^2)/\sigma_\omega^2 & & & \\ & & \ddots & & \\ & & & (1+a^2)/\sigma_\omega^2 & -a/\sigma_\omega^2 \\ & & & -a/\sigma_\omega^2 & 1/\sigma_\omega^2 \end{pmatrix}$$

Kronecker product models

- $\boldsymbol{x} = \{x_{11}, ..., x_{n1}, x_{12}, ..., x_{nT}\}$
- assume

$$\pi(\boldsymbol{x}) \propto (|\boldsymbol{Q1} \otimes \boldsymbol{Q2}|^*)^{1/2} \exp\left(-\frac{1}{2}\boldsymbol{x}^T\{\boldsymbol{Q1} \otimes \boldsymbol{Q2}\}\boldsymbol{x}\right)$$

- $\texttt{dim}(\boldsymbol{Q1} \otimes \boldsymbol{Q2}) = nT$
- $|.|^*$ is the generalized determinant (if needed), [Riebler et al., 2012]

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- a separable space-time in `R-INLA`

```
f(spatial, ### index for space
  model=a.spatial.model,
  group=time, ### index for time
  control.group=list(model=a.one.dim.model))
```

Spacetime interactions

- kronecker product models follows Clayton's rule
- combine $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ available
- WARNING: if $\boldsymbol{Q1}$ or $\boldsymbol{Q2}$ is from an intrinsic model
  - $\boldsymbol{Q1}$ and/or $\boldsymbol{Q2}$ have rank deficiency
  - warning care when main effects are in the model
- the described dynamic model is type IV and uses $\boldsymbol{Q2}$ as AR(1)

Separable SPDE model in INLA

- Define the spatial SPDE model
  - set a mesh, define the SPDE
- Define the temporal model
  - Discrete time [Cameletti et al., 2012]
    - simple AR(1) evolution

Separable SPDE model in INLA

- Define the spatial SPDE model
  - set a mesh, define the SPDE
- Define the temporal model
  - Discrete time [Cameletti et al., 2012]
    - simple AR(1) evolution
  - Continuous time domain [Lindgren and Rue, 2015]
    - Eg. AR(1) on coefficients for 2nd order B-splines

On time knots

from [Lindgren and Rue, 2015]

3.2. Kronecker product models for space-time

The following illustrates the principles for how to set up a Kronecker product space-time model where both space and time are treated continuously, with a Matérn model in space and a stationary AR(1) model on coefficients for 2nd order B-splines in time. Assumed inputs are

```
y           : measurements
station.loc : coordinates for measurement stations
station.id  : for each measurement, which station index?
time        : for each measurement, what time?
```

The time interval for the model is $[0, 100]$.

```
R> knots = seq(0, 100, length = 11)
R> mesh1 = inla.mesh.1d(loc = knots, degree = 2, boundary = "free")
R> mesh2 = inla.mesh.2d(...)
R> spde = inla.spde2.matern(mesh2, alpha = 2, ...)
R> index = inla.spde.make.index("space", n.spde = spde$n.spde, n.group = mesh1$m)
R> formula = y ~ -1 + f(space, model = spde, group = space.group,
+                       control.group = list(model = "ar1"))
R> A = inla.spde.make.A(mesh2, loc = station.loc,
+                       index = station.id, group = time,
+                       group.mesh = mesh1)
R> stack = inla.stack(data = list(y = y), A = list(A), effects = list(index))
```

Outline

Space-time modeling

PM-10 concentration in Piemonte, Italy

A non-separable example

The (linear) measurement equation

- Consider

$$\boldsymbol{y}_{it} = F_{it}'\beta + \boldsymbol{A}_{i(t)}\boldsymbol{x}_t + \epsilon_{it}$$

  - $\boldsymbol{F}_t$ is a matrix of covariates
  - $\boldsymbol{\beta}$ are the fixed effects
  - $\boldsymbol{A}_{(t)}$ picks out the appropriate values of $\boldsymbol{x}_t$
  - $\boldsymbol{\epsilon}_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2\boldsymbol{I})$

The (linear) measurement equation

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  - $\boldsymbol{A}_{(t)}$ picks out the appropriate values of $\boldsymbol{x}_t$
  - $\epsilon_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2 \boldsymbol{I})$
- vector AR(1) process for $\boldsymbol{x}$
$$\boldsymbol{x}_t = \rho \boldsymbol{x}_{t-1} + \boldsymbol{\omega}_t$$
  - $\boldsymbol{\omega_t}$: spatial SPDE model
$$\boldsymbol{\omega}_t \overset{\text{i.i.d.}}{\sim} N(\boldsymbol{0}, \boldsymbol{Q}^{-1}),$$
  - ρ is the time correlation

PM-10 concentration in Piemonte, Italy

Cameletti *et al.* (2011), on `r-inla.org`

- 24 monitoring stations
- Daily data from 10/05 to 03/06

Space model part

- Make the mesh

```
mesh <- inla.mesh.2d(points =NULL,
                     points.domain=borders,
                     offset=c(20, 140),
                     max.edge=c(30, 100))
```

Using the group feature

- Construct a kronecker product model using the group feature

```
formula = y ~ -1 + intercept + WS + HMIX + ... +
   f(field, model=spde,
      group =time,
      control.group=list(model="ar1")
     )
```

- This tells INLA that the observations are grouped in a certain way.
- `control.group` contains the grouping model (`ar1`, `exchangable`, `rw1`, and others) as well as their prior specifications.

Make an $\boldsymbol{A}$ matrix

- Use the `group` argument

```
LocationMatrix = inla.spde.make.A(mesh = mesh,
          loc =dataLoc, group=time, n.group=nT)
```

- data locations in all `group=time` level
- builds an $\boldsymbol{A}$ matrix in an appropriate way

Organising the data

Covariates at the data points, but the latent field only defined their through the A matrix

*We need to make sure that **A** only applies to the random effect.*

```
idx.set <- inla.spde.make.index("mesh.idx",n.field=nmesh,
                  n.group=T)
stack = inla.stack( data = dat,
       A = list(1, LocationMatrix),
       effects = list( list(WS = cov$WS,...),
           c(idx.set,
                     list(intercept=rep(1,mesh$n*nT)))
       )
    )
```

Separable SPDE model (remark)

A spacetime process $z(\boldsymbol{s}, t)$, evolving like

$$
\begin{aligned}
(1 - \gamma_t \frac{\partial}{\partial t})^{\alpha_t} z(\boldsymbol{s}, t) &= \gamma_s^{-1/2} \mathcal{E}(\boldsymbol{s}, t) & (1) \\
(1 - \gamma_{\mathcal{E}} \triangle)^{\alpha_{\mathcal{E}}/2} \mathcal{E}(\boldsymbol{s}, \delta t) &= \mathcal{W}_{\mathcal{E}}(\boldsymbol{s}, \delta t) & (2)
\end{aligned}
$$

where

- $\mathcal{E}(\boldsymbol{s}, t)$ is a spatially correlated process
- $\mathcal{W}_{\mathcal{E}}$ is an unit variance spacetime white noise

is a separable space-time process

Space time heat equation

A spacetime process $z(\boldsymbol{s}, t)$, evolving like

$$(\gamma_t \frac{\partial}{\partial t} - \triangle)^{\alpha_t} z(\boldsymbol{s}, t) = \gamma_s^{-1/2} \mathcal{E}(\boldsymbol{s}, t) \quad (3)$$

$$(1 - \gamma_{\mathcal{E}} \triangle)^{\alpha_{\mathcal{E}}/2} \mathcal{E}(\boldsymbol{s}, \delta t) = \mathcal{W}_{\mathcal{E}}(\boldsymbol{s}, \delta t) \quad (4)$$

where

- $\mathcal{E}(\boldsymbol{s}, t)$ is a spatially correlated process
- $\mathcal{W}_{\mathcal{E}}$ is an unit variance spacetime white noise

is a **non**-separable space-time process

References

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