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Spatial models

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Outline

Smoothing in 2d

Besag variations

Smoothing more

SPDE model

SPDE applications

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Remember 1d: Laplacian for RW1

$x_i - x_{i-1} \sim N(0, 1/(2\tau))$ is the same as

$$\pi(\boldsymbol{x}|\tau) \propto \tau^{(n-1)/2} \exp\left(-\frac{\tau}{2}\sum_{i=2}^{n}(x_i - x_{i-1})^2\right) \quad (1)$$

$$= \tau^{(n-1)/2} \exp\left(-\frac{\tau}{2}\boldsymbol{x}^T \boldsymbol{R}\boldsymbol{x}\right) \quad (2)$$

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when

$$\boldsymbol{R} = \begin{pmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & & & \ddots & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix}$$

$\boldsymbol{R}$ is the Laplacian

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Laplacian (Besag)

randon walk over areas $\pi(x_i|\boldsymbol{x}_{-i},\tau) \sim N(\frac{1}{n_i}\sum_{j\sim i} x_j, \frac{1}{n_i\tau})$

$$\pi(\boldsymbol{x}|\tau) \propto \tau^{(n-1)/2}\exp\left(-\frac{\tau}{2}\sum_{i}^{n}(x_i - \frac{1}{n_i}\sum_{j\sim i} x_j)^2\right) \quad (3)$$

$$= \tau^{(n-1)/2}\exp\left(-\frac{\tau}{2}\sum_{j\sim i}^{n}(x_i - x_j)^2\right) \quad (4)$$

$$= \tau^{(n-1)/2}\exp\left(-\frac{\tau}{2}\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}\right) \quad (5)$$

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when

$$\boldsymbol{R}_{ij} = \begin{cases} n_i & \text{if } i = j \\ -1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases} . \quad (6)$$

$\boldsymbol{R}$ is the Laplacian

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besagproper

$$\pi(x_i|\boldsymbol{x}_{-i},\tau) \sim N(\frac{1}{n_i+d}\sum_{j\sim i} x_j, \frac{1}{\tau(n_i+d)})$$

$$\pi(\boldsymbol{x}|\tau) \propto \left(-\frac{\tau}{2}\boldsymbol{x}^T(\boldsymbol{D}+\boldsymbol{R})\boldsymbol{x}\right) \tag{7}$$

where

- $d > 0$ is an extra parameter
- $\boldsymbol{D} = diag(d, d, ..., d)$
- $\boldsymbol{R}$ as before

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besagproper2

$$\pi(x_i|\boldsymbol{x}_{-i},\tau) \sim N(\textstyle\sum_{j\sim i}(1-\lambda+\frac{\lambda}{n_i})x_j, \frac{1}{\tau[n_i\lambda+(1-\lambda)]}) \text{ for } \lambda \in (0,1)$$

$$\pi(\boldsymbol{x}|\tau) \propto \left(-\frac{\tau}{2}\boldsymbol{x}^T[(1-\lambda)\boldsymbol{I}+\lambda\boldsymbol{R}]\boldsymbol{x}\right) \tag{8}$$

where

— also called Leroux's model

— $\boldsymbol{R}$ as before

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generic1

$\pi(x_i|\boldsymbol{x}_{-i},\tau) \sim N(\frac{\beta}{\lambda_{\max}}\sum_j^n \boldsymbol{C}_{ij}x_j, \frac{1}{\tau})$

$$\pi(\boldsymbol{x}|\tau) \propto \left(-\frac{\tau}{2}\boldsymbol{x}^T(\boldsymbol{I}-\frac{\beta}{\lambda_{\max}}\boldsymbol{C})\boldsymbol{x}\right) \tag{9}$$

where

— $\boldsymbol{C}$ is a structure matrix
  - example: the adjacency matrix

— $\lambda_{\max}$ is the biggest eigenvalue of $\boldsymbol{C}$ to allows $\beta \in [0, 1)$

— conditional variance is not local

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- Besag (and RW1) averages over 1st neighbours
- how to make it smoother?
  - average over 2nd order neighbours? NO
  - use Q^2 as precision? YES!
    - like what RW2 does

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- how to make it smoother?
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    - like what RW2 does

```
n <- 10; (r1 <- INLA:::inla.rw(n, order=1))
```

```
## 10 x 10 sparse Matrix of class "dgTMatrix"
##
##  [1,]  1 -1  .  .  .  .  .  .  .  .
##  [2,] -1  2 -1  .  .  .  .  .  .  .
##  [3,]  . -1  2 -1  .  .  .  .  .  .
##  [4,]  .  . -1  2 -1  .  .  .  .  .
##  [5,]  .  .  . -1  2 -1  .  .  .  .
##  [6,]  .  .  .  . -1  2 -1  .  .  .
##  [7,]  .  .  .  .  . -1  2 -1  .  .
##  [8,]  .  .  .  .  .  . -1  2 -1  .
##  [9,]  .  .  .  .  .  .  . -1  2 -1
## [10,]  .  .  .  .  .  .  .  . -1  1
```

```
(r1 %*% r1)
```

```
## 10 x 10 sparse Matrix of class "dgCMatrix"
##
##  [1,]  2 -3  1  .  .  .  .  .  .  .
##  [2,] -3  6 -4  1  .  .  .  .  .  .
##  [3,]  1 -4  6 -4  1  .  .  .  .  .
##  [4,]  .  1 -4  6 -4  1  .  .  .  .
##  [5,]  .  .  1 -4  6 -4  1  .  .  .
##  [6,]  .  .  .  1 -4  6 -4  1  .  .
##  [7,]  .  .  .  .  1 -4  6 -4  1  .
##  [8,]  .  .  .  .  .  1 -4  6 -4  1
##  [9,]  .  .  .  .  .  .  1 -4  6 -3
## [10,]  .  .  .  .  .  .  .  1 -3  2
```

```
INLA:::inla.rw(n, order=2)
```

```
## 10 x 10 sparse Matrix of class "dgTMatrix"
##
##  [1,]  1 -2  1  .  .  .  .  .  .  .
##  [2,] -2  5 -4  1  .  .  .  .  .  .
##  [3,]  1 -4  6 -4  1  .  .  .  .  .
##  [4,]  .  1 -4  6 -4  1  .  .  .  .
##  [5,]  .  .  1 -4  6 -4  1  .  .  .
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##  [9,]  .  .  .  .  .  .  1 -4  5 -2
## [10,]  .  .                 1 -2  1
```

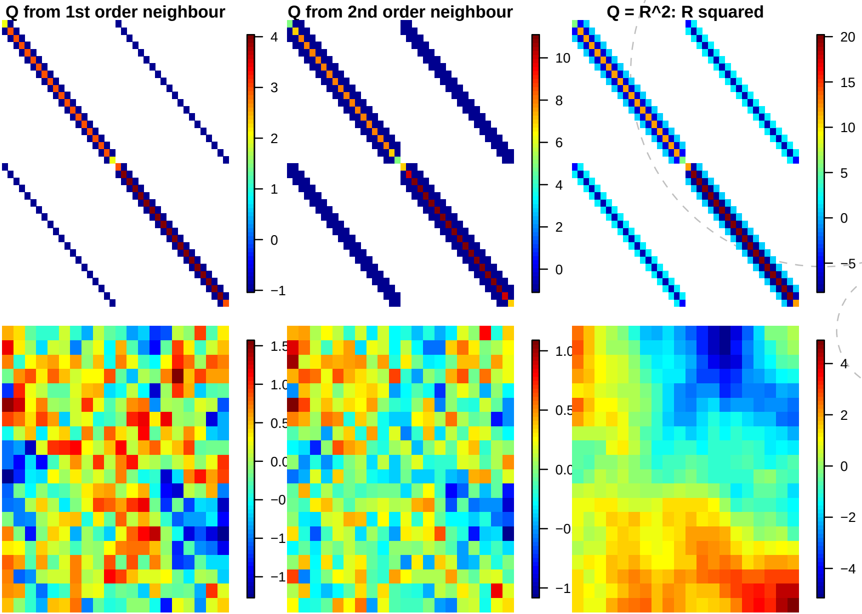

how to make it smoother? $Q_2 = Q_1{}^2$

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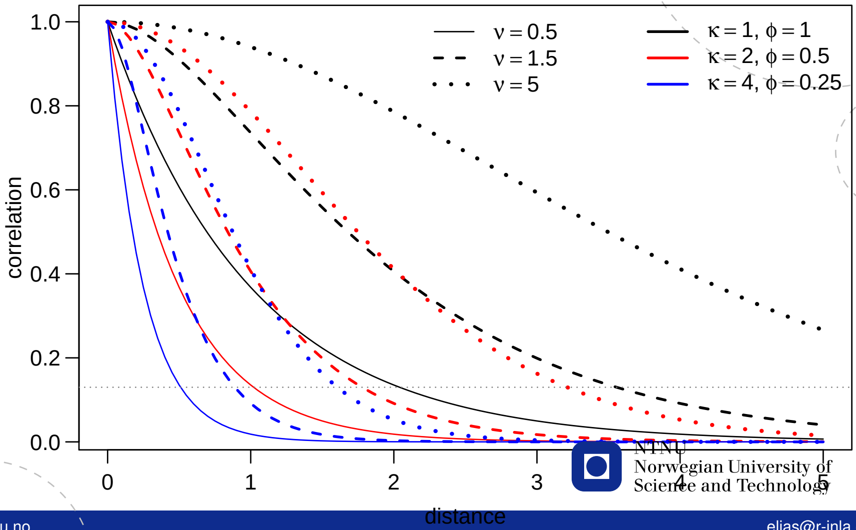
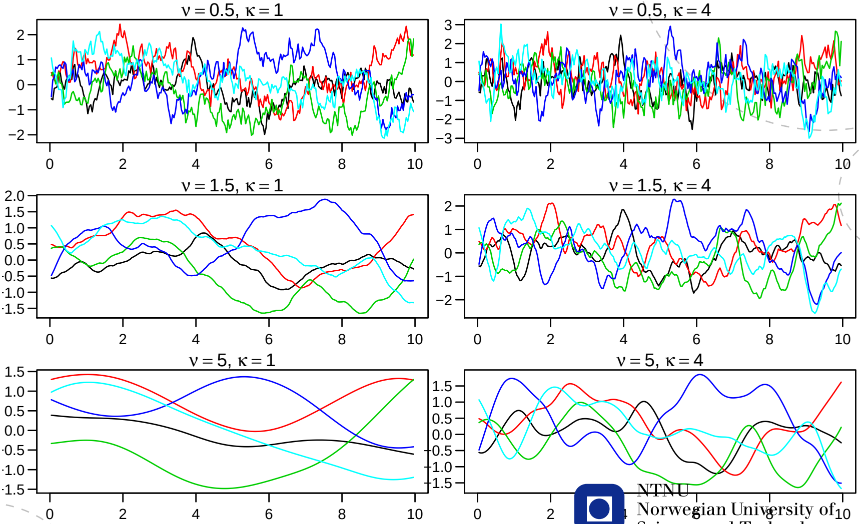
Matérn covariance

$$\Sigma_{ij} = \sigma_x^2 \frac{2^{1-\nu} K_\nu(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)}{\Gamma(\nu)(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)^{-\nu}}, \; \kappa = 1/\phi$$



simulations, 1D, $\sigma_x^2 = 1$



The Stochastic Partial Differential Approach - SPDE

Fields with Matérn covariance are solutions to (SPDE):

$$(\kappa^2 - \Delta)^{\alpha/2}\xi(\boldsymbol{s}) = \tau\mathcal{W}(\boldsymbol{s})$$

- $\kappa > 0$: scale parameter
- $\alpha = \nu + d/2$: smoothness
- Δ is the Laplacian

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial s_i^2}$$

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$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial s_i^2}$$

- When $d = 2$
  - $\alpha = 1$: CAR model
  - $\alpha = 2$: SAR model

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Regular grid, $d = 2$

- $\alpha = 1$: $\boldsymbol{Q}_{1,\kappa} = \boldsymbol{K}_\kappa = \kappa^2 \boldsymbol{C} + \boldsymbol{G}$
- $\boldsymbol{C} = \boldsymbol{I}$, $\boldsymbol{G}$ = Laplacian (4 neighbours)

Laplacian-local pattern:

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

$\boldsymbol{Q}_{1,\kappa}$-local pattern

$$\begin{bmatrix} & -1 & \\ -1 & 4 + \kappa^2 & -1 \\ & -1 & \end{bmatrix}$$

- κ is a scale parameter

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$\boldsymbol{Q}_{1,\kappa}$-local pattern

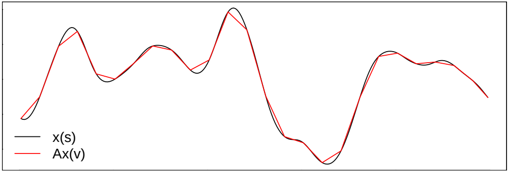
$$\begin{bmatrix} & -1 & \\ -1 & 4+\kappa^2 & -1 \\ & -1 & \end{bmatrix}$$

- κ is a scale parameter
- $\rightarrow$ Sparse precision $\boldsymbol{Q}$ **!!!**
- remember: $(\kappa^2 - \Delta)^{\alpha/2} \xi(\boldsymbol{s}) = \tau \mathcal{W}(\boldsymbol{s})$
- $\rightarrow$ $(\boldsymbol{Q}_{1,\kappa})^{1/2} \xi$ = independent noise
- 'effective' range (0.139) $\approx \sqrt{8\nu/\kappa}$

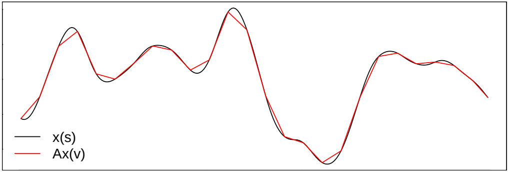
bigger $\alpha \rightarrow$ $\boldsymbol{Q}$ less sparse $\rightarrow$ smoother

- $\alpha = 1$: $\boldsymbol{Q}_{1,\kappa} = \boldsymbol{K}_\kappa = \kappa^2 \boldsymbol{C} + \boldsymbol{G}$
- $\alpha = 2$: $\boldsymbol{Q}_{2,\kappa} = \boldsymbol{K}_\kappa \boldsymbol{C}^{-1} \boldsymbol{K}_\kappa$
- $\alpha = 3, 4, ...$: $\boldsymbol{Q}_{\alpha,\kappa} = \boldsymbol{K}_\kappa \boldsymbol{C}^{-1} \boldsymbol{Q}_{\alpha-2,\kappa} \boldsymbol{C}^{-1} \boldsymbol{K}_\kappa$

Irregular grid → Finite Element Method - FEM → mesh

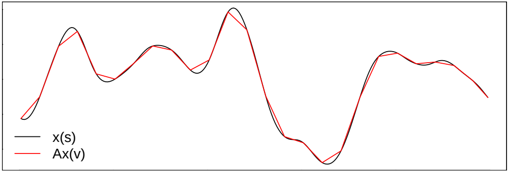


Irregular grid $\rightarrow$ Finite Element Method - FEM $\rightarrow$ mesh



- $\xi(\boldsymbol{s}) \approx \sum_{k=1}^{m} \psi_k(\boldsymbol{s}) w_k = \boldsymbol{A}\xi(\boldsymbol{v},$
- ψ_k: basis functions,
- w_k: weights

Irregular grid → Finite Element Method - FEM → mesh



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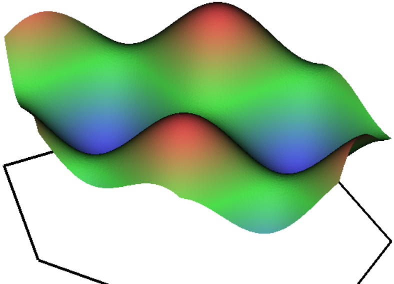

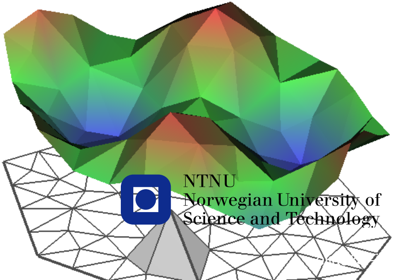

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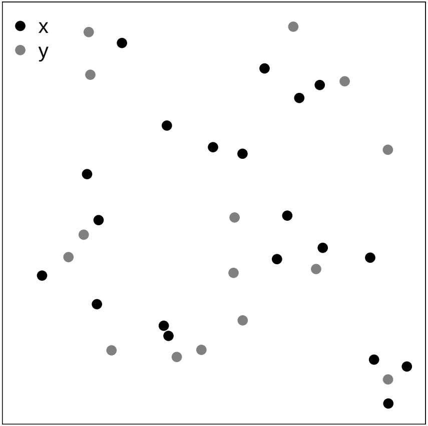
Bivariate and misaligned



- $x(\boldsymbol{s}_j) = x_j$: covariate at n_x locations $\boldsymbol{s}_j$
- $y(\boldsymbol{s}_i) = y_j$: response at n_y locations $\boldsymbol{s}_i$
- can be partially or totally misalined

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Point-process: log-Cox

- regular grid free approach
- $\lambda(\boldsymbol{s})$: intensity function
- $\log(\lambda(\boldsymbol{s})) = \xi(\boldsymbol{s})$
- $\log(\pi(y|\lambda)) =$

$$|\Omega| - \int_{\Omega} e^{\xi(\boldsymbol{s})} d\boldsymbol{s} + \sum_{i=1}^{n} \xi(\boldsymbol{s}_i)$$

$$\approx c - \boldsymbol{w}^T e^{\xi(\boldsymbol{v})} + \mathbf{1}^T \boldsymbol{A} \xi(\boldsymbol{v}))$$

$\boldsymbol{w}$ is $\tilde{\boldsymbol{C}}_{ii}$ for non-boundary $\boldsymbol{v}_i$

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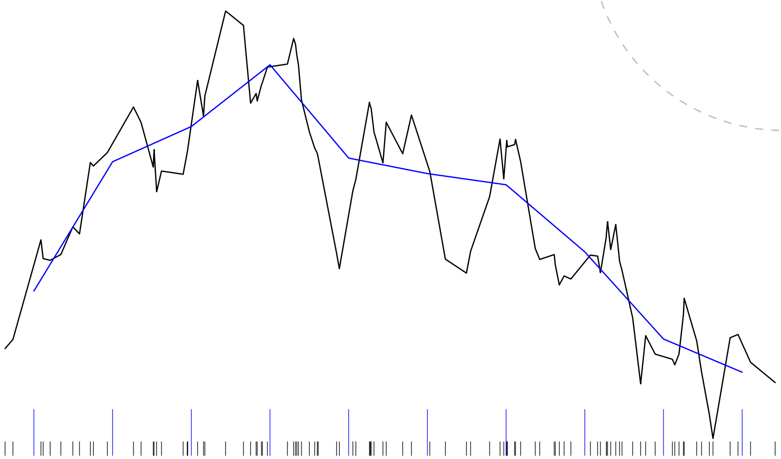
Preferential sampling

— joint model for locations and marks
— test if sampling locations are preferential
— log-Cox model for locations

1d: Continuous time-series



→ lowering time dimension

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Non-stationary

- parametric way
- basis/covariates $\boldsymbol{B}$
- $\log(\tau_i) = \boldsymbol{B}_0^{(\tau)} + \sum_{j=1}^{p} \boldsymbol{B}_{i,j}^{(\tau)} \theta_j^{(\tau)}$
- $\log(\kappa_i) = \boldsymbol{B}_0^{(\kappa)} + \sum_{j=1}^{p} \boldsymbol{B}_{i,j}^{(\kappa)} \theta_j^{(\kappa)}$

```
 spde <- inla.spde2.matern( mesh=..., B.tau=...,
B.kappa=..., ...)
```

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